Asymptotic analysis of the Hermite polynomials from their differential-difference equation

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Abstract

We analyze the Hermite polynomials $H_n(x)$ and their zeros asymptotically, as $n \to \infty$. We obtain asymptotic approximations from the differential-difference equation which they satisfy, using the ray method. We give numerical examples showing the accuracy of our formulas.

Keywords: Hermite polynomials, asymptotic analysis, ray method, orthogonal polynomials, differential-difference equations, discrete WKB method.
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1 Introduction

It would be difficult to find a more ubiquitous polynomial family than the Hermite polynomials $H_n(x)$, defined by the Rodrigues formula

$$H_n(x) = (-1)^n \exp\left(x^2\right) \frac{d^n}{dx^n} \exp\left(-x^2\right), \quad n = 0, 1, 2, \ldots.$$

They appear in several problems of mathematical physics [23], the most important probably being the solution of the Schrödinger equation [6], [14]. Being the limiting case of several families of classical orthogonal polynomials [16], they are of fundamental importance in asymptotic analysis [24], [33].

The Hermite polynomials satisfy the orthogonality condition

$$\int_{-\infty}^{\infty} e^{-x^2} H_m(x) H_n(x) dx = \sqrt{\pi} 2^n n! \delta_{mn},$$

the differential-difference equation

$$H_{n+1} + H_n' = 2x H_n, \tag{1}$$

and the reflection formula

$$H_n(-x) = (-1)^n H_n(x). \tag{2}$$

The zeros of the Hermite polynomials have several applications, notably in Gauss' quadrature formula for numerical integration [13], [29], [30]. Several properties and their asymptotic behavior were studied in [1], [2], [3], [4], [7], [26], [28], [31] and [32].

The asymptotic behavior of $H_n(x)$ was studied by M. Plancherel and W. Rotach in [27] using the method that now bears their name. F. W. J. Olver [25] obtained asymptotic expansions for the Hermite polynomials as a consequence of his WKB analysis of the differential equation satisfied by the Parabolic Cylinder function $D_\nu(z)$, related to $H_n(x)$ by

$$H_n(x) = 2^{\frac{n}{2}} \exp\left(\frac{x^2}{2}\right) D_n\left(\sqrt{2}x\right).$$

A similar analysis using perturbation techniques was carried on by A. Voznyuk in [34].

As an application of the results from his doctoral thesis on the multiplication-interpolation method, L. Heflinger [15] established asymptotic series for the Hermite polynomials. In [39], M. Wyman derived asymptotic formulas for $H_n(x)$ based on one of their integral representations.

In this paper we shall take a different approach and analyze the differential-difference equations that the Hermite polynomials satisfy (1) using the techniques presented in [10]. A similar method (which we may call the discrete WKB method) has been applied to the solution of difference equations [5], [8], [12], [38] and it is currently being extended [11], [35], [36], [37], to include difference equations with turning points. Another type of analysis, based on perturbation techniques, was considered by C. Lange and R. Miura in [17], [18], [19], [20], [21], and [22].

2 Asymptotic analysis

We consider the approximation

$$H_n(x) \sim \exp\left[f(x,n) + g(x,n)\right], \quad n \to \infty \tag{3}$$

where

$$g = o(f), \quad n \to \infty. \tag{4}$$

Note that since $H_0(x) = 1$, we must have

$$f(x,0) = 0 \tag{5}$$

and

$$g(x,0) = 0. \tag{6}$$

Using (3) in (1), we have

$$\begin{aligned} &\exp\left(f + \frac{\partial f}{\partial n} + \frac{1}{2}\frac{\partial^2 f}{\partial n^2} + g + \frac{\partial g}{\partial n}\right) \\ &+ \left(\frac{\partial f}{\partial x} + \frac{\partial g}{\partial x}\right)\exp\left(f+g\right) = 2x\exp\left(f+g\right), \end{aligned} \tag{7}$$

where we have used

$$f(x,n+1) = f(x,n) + \frac{\partial f}{\partial n}(x,n) + \frac{1}{2}\frac{\partial^2 f}{\partial n^2}(x,n) + \cdots .$$

Simplifying (7) and taking (4) into account we obtain, to leading order,

$$\exp\left(\frac{\partial f}{\partial n}\right)+\frac{\partial f}{\partial x}=2x. \tag{8}$$

Using (8) in (7) we get

$$\exp\left(\frac{1}{2}\frac{\partial^2 f}{\partial n^2}+\frac{\partial g}{\partial n}\right)+\frac{\partial g}{\partial x}\exp\left(-\frac{\partial f}{\partial n}\right)=1,$$

or, to leading order,

$$\frac{1}{2}\frac{\partial^2 f}{\partial n^2}+\frac{\partial g}{\partial n}+\frac{\partial g}{\partial x}\exp\left(-\frac{\partial f}{\partial n}\right)=0. \tag{9}$$

2.1 The ray expansion

To solve (8) we use the method of characteristics, which we briefly review. Given the first order partial differential equation

$$F\left(x,n,f,p,q\right)=0,$$

where

$$p=\frac{\partial f}{\partial x},\quad q=\frac{\partial f}{\partial n},$$

we search for a solution $f(x,n)$ by solving the system of "characteristic equations"

$$\begin{aligned}
&\frac{dx}{dt}=\frac{\partial F}{\partial p},\quad \frac{dn}{dt}=\frac{\partial F}{\partial q},\\
&\frac{dp}{dt}=-\frac{\partial F}{\partial x}-p\frac{\partial F}{\partial f},\quad \frac{dq}{dt}=-\frac{\partial F}{\partial n}-q\frac{\partial F}{\partial f},\\
&\frac{df}{dt}=p\frac{\partial F}{\partial p}+q\frac{\partial F}{\partial q},
\end{aligned}$$

where we now consider $\{x,n,f,p,q\}$ to all be functions of the variables t and s.

For (8), we have

$$F\left(x,n,f,p,q\right)=e^{q}+p-2x \tag{10}$$

and therefore the characteristic equations are

$$\frac{dx}{dt} = 1, \quad \frac{dn}{dt} = e^q, \quad \frac{dp}{dt} = 2, \quad \frac{dq}{dt} = 0, \tag{11}$$

and

$$\frac{df}{dt} = p + qe^q. \tag{12}$$

Solving (11) subject to the initial conditions

$$x(0,s) = s, \quad n(0,s) = 0, \quad q(0,s) = A(s), \tag{13}$$

we obtain

$$x = t + s, \quad n = te^A, \quad p = 2t + 2s - e^A, \quad q = A, \tag{14}$$

where we have used

$$0 = F|_{t=0} = e^A + p(0,s) - 2s.$$

From (5) and (13) we have

$$f(0,s) = 0, \tag{15}$$

which implies

$$\begin{aligned} 0 = \frac{d}{ds} f(0,s) &= \left[\frac{\partial f}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial f}{\partial n}\frac{\partial n}{\partial s} \right]_{t=0} \\ &= p(0,s) \times 1 + q(0,s) \times 0 = 2s - e^A. \end{aligned}$$

Thus, $A(s) = \ln(2s)$ and (14) becomes

$$x = t + s, \quad n = 2ts, \quad p = 2t, \quad q = \ln(2s), \tag{16}$$

with $t \geq 0$ and $s > 0$. Since $s > 0$, we shall consider only the region $x > 0$ for now. Using (16) in (12) and taking (15) into account, we obtain

$$f(t,s) = t^2 + 2s\ln(2s)t. \tag{17}$$

Solving for t and s in terms of x and n in (16), we get

$$t = \frac{x}{2} \pm \frac{1}{2}\sigma, \quad s = \frac{x}{2} \mp \frac{1}{2}\sigma \tag{18}$$

with

$$\sigma = \sqrt{x^2 - 2n}. \tag{19}$$

For σ to be a real number, we shall impose the condition $x > \sqrt{2n}$. Since (for a fixed value of n) we have $t \to 0$ as $x \to \infty$, we consider the solution

$$t = \frac{x}{2} - \frac{1}{2}\sigma, \quad s = \frac{x}{2} + \frac{1}{2}\sigma. \tag{20}$$

Replacing (20) in (17) we obtain

$$f(x,n) = \frac{x^2 - \sigma x - n}{2} + n \ln (x + \sigma), \quad x > \sqrt{2n}. \tag{21}$$

We shall now find $g(x,n)$. Using (21) in (9), we get

$$-\frac{1}{2\sigma (x+\sigma)} + \frac{\partial g}{\partial n} + \frac{\partial g}{\partial x}\frac{1}{(x+\sigma)} = 0,$$

or

$$(x+\sigma)\frac{\partial g}{\partial n} + \frac{\partial g}{\partial x} = \frac{1}{2\sigma}. \tag{22}$$

Solving (22), we obtain

$$g(x,n) = \frac{1}{2}\ln\left(-2\frac{x^2 - n + x\sigma}{x^2 - 2n + x\sigma}\right) + C(x+\sigma),$$

where $C(x)$ is a function to be determined. Imposing the condition (6), we have

$$0 = g(x,0) = \frac{1}{2}\ln(-2) + C(2x).$$

Thus,

$$g(x,n) = \frac{1}{2}\ln\left[\frac{1}{2}\left(\frac{x}{\sigma} + 1\right)\right]. \tag{23}$$

We summarize our results in the following theorem.

Theorem 1 *In the region $x > \sqrt{2n}$, the Hermite polynomials admit the asymptotic representation*

$$H_n(x) \sim \Phi_1(x,n) = \exp\left[\frac{x^2 - \sigma x - n}{2} + n\ln(\sigma + x)\right] \times \sqrt{\frac{1}{2}\left(1 + \frac{x}{\sigma}\right)}, \quad n \to \infty, \tag{24}$$

where $\sigma(x,n)$ was defined in (19).

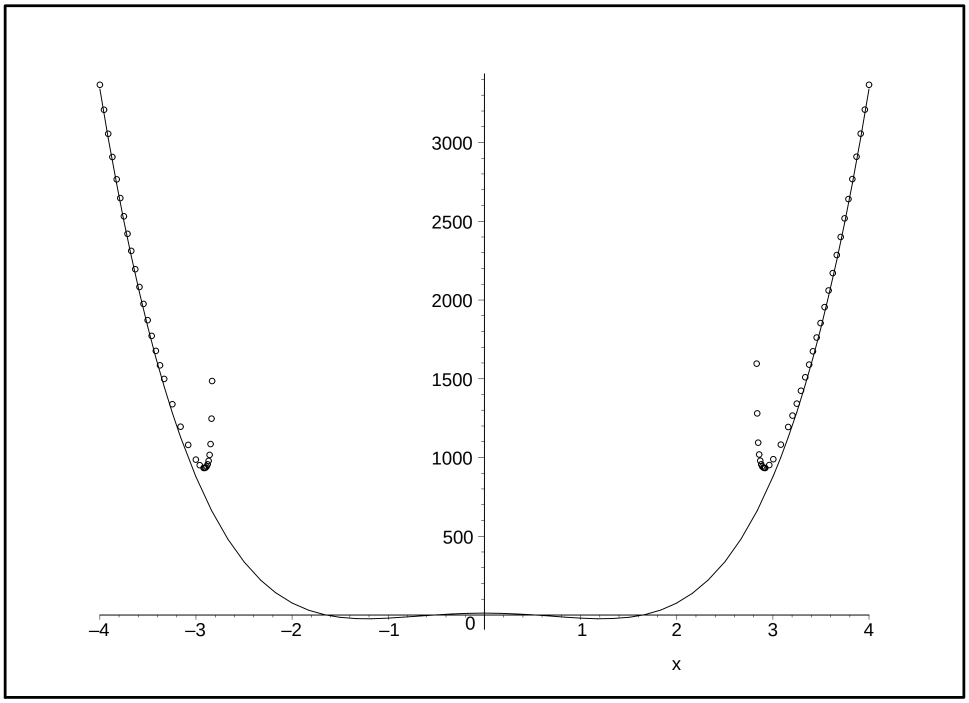

Figure 1: A comparison of $H_4(x)$ (solid curve) and the asymptotic approximations $\Phi_1(x,4)$ and $\Phi_2(x,4)$ (ooo).

Using the reflection formula (2) we can extend our result to the region $-x > \sqrt{2n}$ and obtain:

Corollary 2 *In the region $x < -\sqrt{2n}$, the Hermite polynomials admit the asymptotic representation*

$$H_n(x) \sim \Phi_2(x,n) = (-1)^n \exp\left[\frac{x^2+\sigma x - n}{2} + n\ln(\sigma - x)\right] \times \sqrt{\frac{1}{2}\left(1-\frac{x}{\sigma}\right)}, \quad n \to \infty. \tag{25}$$

To illustrate the accuracy of our results, in Figure 1 we graph $H_4(x)$ and the asymptotic approximations $\Phi_1(x,4)$ and $\Phi_2(x,4)$.

2.2 The transition layer

We shall now find an asymptotic approximation for $|x| \approx \sqrt{2n}$. We will consider the case $x \approx \sqrt{2n}$ and find the corresponding result for $x \approx -\sqrt{2n}$ by using (2). From (24) we have

$$\Phi_1(x,n) \sim \exp\left[\frac{n}{2}\ln(2n) - \frac{3}{2}n + \sqrt{2n}x\right], \quad x \longrightarrow \sqrt{2n}^+.$$

We define the function $G_n(x)$ by

$$H_n(x) = \exp\left[\frac{n}{2}\ln(2n) - \frac{3}{2}n + \sqrt{2n}x\right] G_n(x). \tag{26}$$

Using (26) in (1) we get

$$\begin{gathered}\exp\left[\frac{n+1}{2}\ln(2n+2) - \frac{n}{2}\ln(2n) - \frac{3}{2} + \left(\sqrt{2(n+1)} - \sqrt{2n}\right)x\right] G_{n+1} \\ +\sqrt{2n}G_n + G_n' = 2xG_n(x).\end{gathered} \tag{27}$$

We introduce the stretch variable $\beta > 0$ defined by

$$x = \sqrt{2n} + \frac{\beta}{n^{\frac{1}{6}}} \tag{28}$$

and the function $\Lambda(\beta)$ defined by

$$G_n(x) = \Lambda\left[\left(x - \sqrt{2n}\right) n^{\frac{1}{6}}\right]. \tag{29}$$

From (28) we have

$$\begin{gathered}\exp\left[\frac{n+1}{2}\ln(2n+2) - \frac{n}{2}\ln(2n) - \frac{3}{2} + \left(\sqrt{2(n+1)} - \sqrt{2n}\right)x\right] \\ \sim \sqrt{2n} + \beta n^{-\frac{1}{6}}, \quad n \to \infty.\end{gathered} \tag{30}$$

Using (28) in (29) we obtain

$$\begin{aligned}G_{n+1}(x) &= \Lambda\left[\left(\sqrt{2n} - \sqrt{2n+1} + \frac{\beta}{n^{\frac{1}{6}}}\right)(n+1)^{\frac{1}{6}}\right] \\ &\sim \Lambda(\beta) - \frac{1}{\sqrt{2}}\Lambda'(\beta)\, n^{-\frac{1}{3}} + \frac{1}{4}\Lambda''(\beta)\, n^{-\frac{2}{3}}, \quad n \to \infty,\end{aligned} \tag{31}$$

and

$$\sqrt{2n}G_n + G_n' - 2xG_n(x) = -\sqrt{2n}\Lambda(\beta) + \Lambda'(\beta)\, n^{\frac{1}{6}} - 2\beta\Lambda(\beta)\, n^{-\frac{1}{6}}. \qquad (32)$$

Using (30), (31) and (32) in (27) we obtain, to leading order, the Airy equation

$$\Lambda''(\beta) = 2\sqrt{2}\beta\Lambda(\beta). \qquad (33)$$

Thus,

$$\Lambda(\beta) = C_1 \operatorname{Ai}\left(\sqrt{2}\beta\right) + C_2 \operatorname{Bi}\left(\sqrt{2}\beta\right), \qquad (34)$$

where $\operatorname{Ai}(\cdot)$ and $\operatorname{Bi}(\cdot)$ denote the Airy functions and C_1, C_2 are to be determined. Replacing (28) and (34) in (26) we have

$$H_n(x) \sim \exp\left[\frac{n}{2}\ln(2ne) + \sqrt{2}\beta n^{\frac{1}{3}}\right]\left[C_1 \operatorname{Ai}\left(\sqrt{2}\beta\right) + C_2 \operatorname{Bi}\left(\sqrt{2}\beta\right)\right]. \qquad (35)$$

To find C_1, C_2 we shall match (35) with (24). Using (28) in (24) we get

$$\Phi_1(x,n) \sim \exp\left[\frac{n}{2}\ln(2ne) + \sqrt{2}\beta n^{\frac{1}{3}} - \frac{2^{\frac{7}{4}}}{3}\beta^{\frac{3}{2}}\right] 2^{-\frac{5}{8}}\beta^{-\frac{1}{4}} n^{\frac{1}{6}}, \qquad (36)$$

as $\beta \to 0$. Using (28) and the well known asymptotic expansions of the Airy functions

$$\begin{aligned} \operatorname{Ai}(x) &\sim \frac{1}{2\sqrt{\pi}} \exp\left(-\frac{2}{3}x^{\frac{3}{2}}\right) x^{-\frac{1}{4}}, \quad x \to \infty \\ \operatorname{Bi}(x) &\sim \frac{1}{\sqrt{\pi}} \exp\left(\frac{2}{3}x^{\frac{3}{2}}\right) x^{-\frac{1}{4}}, \quad x \to \infty, \end{aligned}$$

in (35) we have

$$\exp\left[\frac{n}{2}\ln(2n) - \frac{3}{2}n + \sqrt{2n}x\right] = \exp\left\{\frac{n}{2}\left[1 + \ln(2n)\right] + \sqrt{2}\beta n^{\frac{1}{3}}\right\} \qquad (37)$$

and

$$C_1 \operatorname{Ai}\left(\sqrt{2}\beta\right) + C_2 \operatorname{Bi}\left(\sqrt{2}\beta\right) \sim \qquad (38)$$
$$\frac{C_1}{\sqrt{\pi}2^{\frac{9}{8}}\beta^{\frac{1}{4}}} \exp\left(-\frac{2^{\frac{7}{4}}}{3}\beta^{\frac{3}{2}}\right) + \frac{C_2}{\sqrt{\pi}2^{\frac{1}{8}}\beta^{\frac{1}{4}}} \exp\left(\frac{2^{\frac{7}{4}}}{3}\beta^{\frac{3}{2}}\right), \quad \beta \to \infty.$$

Matching (36) to (37) and (38), we conclude that

$$C_1 = \sqrt{2\pi} n^{\frac{1}{6}}, \quad C_2 = 0. \tag{39}$$

This completes the analysis. Combining the results above, we have the following result:

Theorem 3 *For $x \approx \sqrt{2n}$, the Hermite polynomials have the asymptotic representation*

$$\begin{aligned} H_n(x) \sim \Phi_3(x,n) = \exp\left[\frac{n}{2}\ln(2n) - \frac{3}{2}n + \sqrt{2n}x\right] \\ \times \sqrt{2\pi} n^{\frac{1}{6}} \operatorname{Ai}\left[\sqrt{2}\left(x - \sqrt{2n}\right) n^{\frac{1}{6}}\right], \quad n \to \infty. \end{aligned} \tag{40}$$

Use of the reflection formula (2) provides the corresponding result for $x \approx -\sqrt{2n}$.

Corollary 4 *For $x \approx -\sqrt{2n}$, the Hermite polynomials have the asymptotic representation*

$$\begin{aligned} H_n(x) \sim \Phi_4(x,n) = (-1)^n \exp\left[\frac{n}{2}\ln(2n) - \frac{3}{2}n - \sqrt{2n}x\right] \\ \times \sqrt{2\pi} n^{\frac{1}{6}} \operatorname{Ai}\left[-\sqrt{2}\left(x + \sqrt{2n}\right) n^{\frac{1}{6}}\right], \quad n \to \infty. \end{aligned} \tag{41}$$

2.3 The oscillatory region

We now study the region bounded by the curve $n = \frac{x^2}{2}$, where the zeros of $H_n(x)$ are located. In this region, the solution is a linear combination of (24) and (25)

$$H_n(x) \sim \Phi_5(x,n) \equiv K_1 \Phi_1(x,n) + K_2 \Phi_2(x,n), \quad n \to \infty$$

with $|x| < \sqrt{2n}$ and K_1,K_2 are constants to be determined. We shall require $\Phi_5(x,n)$ to match $\Phi_3(x,n)$ asymptotically in the local variable β, i.e., it must satisfy the limiting condition

$$\lim_{\beta \to 0} \Phi_5(\beta, n) = \lim_{\beta \to -\infty} \Phi_3(\beta, n).$$

Writing (40) in terms of β, we have

$$\Phi_3(\beta,n) = \exp\left[\frac{n}{2}\ln(2ne) + \sqrt{2}\beta n^{\frac{1}{3}}\right]\sqrt{2\pi}n^{\frac{1}{6}}\,\mathrm{Ai}\left(\sqrt{2}\beta\right). \tag{42}$$

Using the asymptotic formula

$$\mathrm{Ai}(x) \sim \frac{1}{\sqrt{\pi}}\sin\left[\frac{2}{3}(-x)^{\frac{3}{2}} + \frac{\pi}{4}\right](-x)^{-\frac{1}{4}}, \quad x \to -\infty$$

in (42) we get

$$\Phi_3(\beta,n) \sim \exp\left[\frac{n}{2}\ln(2ne) + \sqrt{2}\beta n^{\frac{1}{3}}\right]2^{\frac{3}{8}}n^{\frac{1}{6}}$$
$$\times\sin\left[\frac{1}{3}2^{\frac{7}{4}}(-\beta)^{\frac{3}{2}} + \frac{\pi}{4}\right](-\beta)^{-\frac{1}{4}}, \quad \beta \to -\infty,$$

which can be rewritten as

$$\Phi_3(\beta,n) \sim 2^{-\frac{5}{8}}n^{\frac{1}{6}}\beta^{-\frac{1}{4}}\exp\left[\frac{n}{2}\ln(2ne) + \sqrt{2}\beta n^{\frac{1}{3}}\right] \tag{43}$$
$$\times\left[\exp\left(-\frac{1}{3}2^{\frac{7}{4}}\beta^{\frac{3}{2}}\right) + \mathrm{i}\exp\left(\frac{1}{3}2^{\frac{7}{4}}\beta^{\frac{3}{2}}\right)\right], \quad \beta \to -\infty.$$

Using (28) in (24), we have

$$\Phi_1(\beta,n) \sim \exp\left[\frac{n}{2}\ln(2ne) + \sqrt{2}\beta n^{\frac{1}{3}} - \frac{1}{3}2^{\frac{7}{4}}\beta^{\frac{3}{2}}\right]2^{-\frac{5}{8}}n^{\frac{1}{6}}\beta^{-\frac{1}{4}}, \quad \beta \to 0. \tag{44}$$

Similarly, using (28) in (25), we obtain

$$\Phi_2(\beta,n) \sim \exp\left[\frac{n}{2}\ln(2ne) + \sqrt{2}\beta n^{\frac{1}{3}} + \frac{1}{3}2^{\frac{7}{4}}\beta^{\frac{3}{2}}\right]2^{-\frac{5}{8}}n^{\frac{1}{6}}\beta^{-\frac{1}{4}}\mathrm{i}, \quad \beta \to 0, \tag{45}$$

where we have used

$$(-1)^n\exp\left[\frac{x^2+\sigma x-n}{2} + n\ln(\sigma - x)\right] = \exp\left[\frac{x^2+\sigma x-n}{2} + n\ln(x-\sigma)\right].$$

Comparing (43) with (44) and (45) we conclude that $K_1 = 1 = K_2$ and therefore

$$\Phi_5(x,n) = \Phi_1(x,n) + \Phi_2(x,n). \tag{46}$$

Since $-\sqrt{2n} < x < \sqrt{2n}$, we set

$$x = \sqrt{2n} \sin(\theta), \quad -\frac{\pi}{2} < \theta < \frac{\pi}{2}. \tag{47}$$

Using (47) in (19), we have

$$\sigma = \sqrt{2n} \cos(\theta)\, \mathrm{i}. \tag{48}$$

Replacing (48) in (24), we get

$$\exp\left[\frac{x^2 - \sigma x - n}{2} + n \ln(\sigma + x)\right] = \tag{49}$$
$$\exp\left\{\frac{n}{2}\left[\ln(2n) - \cos(2\theta)\right] - n\left[\frac{1}{2}\sin(2\theta) + \theta - \frac{\pi}{2}\right]\mathrm{i}\right\}$$

and

$$\sqrt{\frac{1}{2}\left(1 + \frac{x}{\sigma}\right)} = \frac{\exp\left(-\frac{\theta}{2}\mathrm{i}\right)}{\sqrt{2\cos(\theta)}}. \tag{50}$$

Similarly, replacing (48) in (25), we obtain

$$(-1)^n \exp\left[\frac{x^2 + \sigma x - n}{2} + n \ln(\sigma - x)\right] = \tag{51}$$
$$\exp\left\{\frac{n}{2}\left[\ln(2n) - \cos(2\theta)\right] + n\left[\frac{1}{2}\sin(2\theta) + \theta - \frac{\pi}{2}\right]\mathrm{i}\right\}$$

and

$$\sqrt{\frac{1}{2}\left(1 - \frac{x}{\sigma}\right)} = \frac{\exp\left(\frac{\theta}{2}\mathrm{i}\right)}{\sqrt{2\cos(\theta)}}. \tag{52}$$

Using (49)–(52) in (46), we have

$$\Phi_5\left[\sqrt{2n}\sin(\theta), n\right] = \sqrt{\frac{2}{\cos(\theta)}} \exp\left\{\frac{n}{2}\left[\ln(2n) - \cos(2\theta)\right]\right\} \tag{53}$$
$$\times \cos\left\{n\left[\frac{1}{2}\sin(2\theta) + \theta - \frac{\pi}{2}\right] + \frac{\theta}{2}\right\}, \quad -\frac{\pi}{2} < \theta < \frac{\pi}{2}.$$

Thus, we have proved the following:

Theorem 5 *In the region $|x| < \sqrt{2n}$, the Hermite polynomials have the asymptotic representation*

$$H_n\left[\sqrt{2n}\sin(\theta)\right] \sim \sqrt{\frac{2}{\cos(\theta)}}\exp\left\{\frac{n}{2}\left[\ln(2n) - \cos(2\theta)\right]\right\} \tag{54}$$
$$\times\cos\left\{n\left[\frac{1}{2}\sin(2\theta) + \theta - \frac{\pi}{2}\right] + \frac{\theta}{2}\right\}, \quad n \to \infty,$$

with $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$.

In Figure 2 we graph

$$H_n\left[\sqrt{2n}\sin(\theta)\right]\exp\left\{-\frac{n}{2}\left[\ln(2n) - \cos(2\theta)\right]\right\}$$

and

$$\sqrt{\frac{2}{\cos(\theta)}}\cos\left\{n\left[\frac{1}{2}\sin(2\theta) + \theta - \frac{\pi}{2}\right] + \frac{\theta}{2}\right\},$$

with $n = 20$. We only include the range $0 \leq \theta < \frac{\pi}{2}$, since both functions are even.

The same results obtained in this section were derived in [9] using a different method, based on the limit relation between the Charlier and Hermite polynomials [16].

3 Zeros

We shall now find asymptotic formulas for the zeros of the Hermite polynomials using the results from the previous section. Let's denote by $\zeta_1^n > \zeta_2^n > \cdots > \zeta_n^n$ the zeros of $H_n(x)$, enumerated in decreasing order. Then, it follows from (54) that

$$\zeta_k^n \sim \sqrt{2n}\sin(\tau_k^n), \quad n \to \infty \tag{55}$$

where τ_k^n is a solution of the equation

$$n\left[\frac{1}{2}\sin(2\tau_k^n) + \tau_k^n - \frac{\pi}{2}\right] + \frac{\tau_k^n}{2} = (1-2k)\frac{\pi}{2}, \quad 1 \leq k \leq n. \tag{56}$$

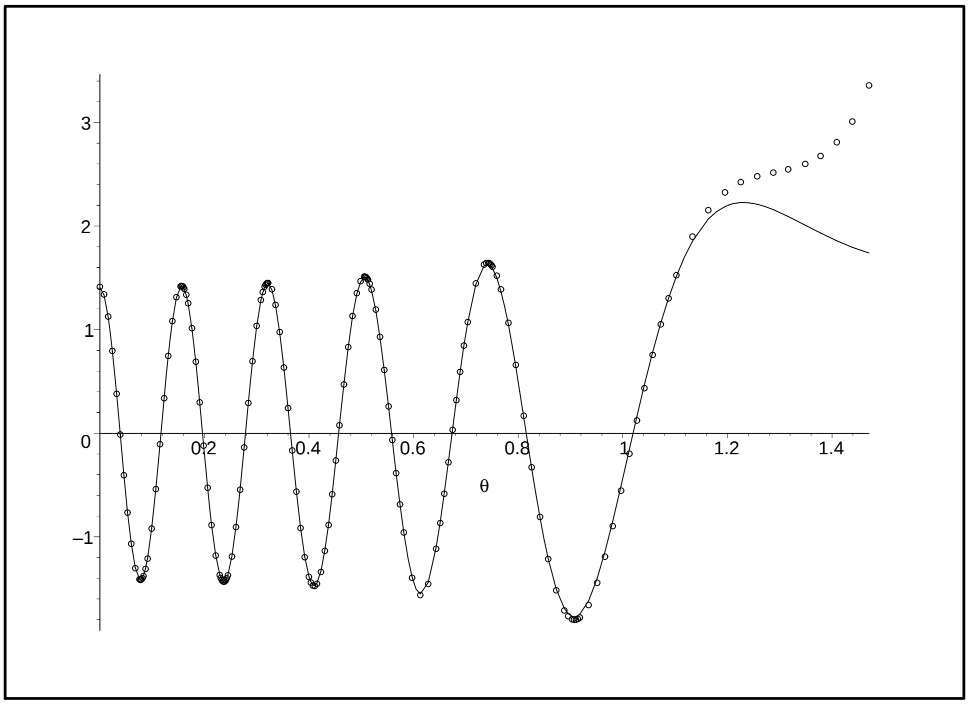

Figure 2: A comparison of the exact (solid curve) and asymptotic (ooo) values of $H_{20}\,(x)$ in the oscillatory region.

Solving (56) numerically and using (55) we get very good approximations of ζ_k^n. One could also solve (56) exactly (as we did in [9]) and obtain a Kapteyn series expansion for τ_k^n

$$\tau_k^n = \frac{\pi}{2} - \frac{\pi}{2}(4k-1)N^{-1} - \sum_{j=1}^{\infty} \frac{1}{j} \mathrm{J}_j \left[\left(1 - N^{-1}\right) j \right] \sin\left(\frac{4k-1}{N} j\pi \right), \quad (57)$$

where $N = 2n+1$ and $\mathrm{J}_j(\cdot)$ denotes the Bessel function of the first kind. However, (57) is difficult to analyze asymptotically. Hence, we will take a different approach and find an approximation for τ_k^n from (56) trough perturbation techniques.

We will consider two cases: $k = O(1)$ which corresponds to the largest zeros of $H_n(x)$ and $k = O\left(\frac{n}{2}\right)$, related to the zeros close to $x = 0$.

3.1 Case I: $k = O(1)$

Replacing

$$\tau_k^n = \frac{\pi}{2} - \sum_{i \geq 1} a_i(k)\, n^{-\frac{i}{3}} \quad (58)$$

in (56) we obtain, as $n \to \infty$

$$\begin{aligned}
a_1 &= \frac{1}{2}\kappa^{\frac{1}{3}}, \quad a_2 = -\frac{1}{2}\kappa^{-\frac{1}{3}}, \quad a_3 = \frac{\kappa}{120}, \quad a_4 = -\frac{\kappa^{-\frac{5}{3}}}{30}\left(\kappa^2 - 5\right) \\
a_5 &= \frac{\kappa^{-\frac{7}{3}}}{8400}\left(3\kappa^4 + 350\kappa^2 + 1400\right), \quad a_6 = -\frac{43}{16800}\kappa, \\
a_7 &= \frac{\kappa^{-\frac{11}{3}}}{50400}(\kappa^6 + 350\kappa^4 - 980\kappa^2 - 11200), \\
a_8 &= -\frac{\kappa^{-\frac{13}{3}}}{63000}(13\kappa^6 + 475\kappa^4 + 1400\kappa^2 + 17500), \\
a_9 &= \frac{59}{67200}\kappa + \frac{43}{34496000}\kappa^3, \\
a_{10} &= -\frac{\kappa^{-\frac{17}{3}}}{1397088000}(23817\kappa^8 + 2608760\kappa^6 - 4592280\kappa^4 \\
&\quad - 51744000\kappa^2 - 664048000),
\end{aligned} \quad (59)$$

with

$$\kappa(k) = 3\pi(4k-1). \quad (60)$$

Using (58)-(59) in (55), we get

$$\begin{aligned}\zeta_k^n \sim \sqrt{2}\Bigg(& n^{\frac{1}{2}} - \frac{\kappa^{\frac{2}{3}}}{8} n^{-\frac{1}{6}} + \frac{1}{4} n^{-\frac{1}{2}} - \frac{\kappa^2+80}{640\kappa^{\frac{2}{3}}} n^{-\frac{5}{6}} \\ & - \frac{11\kappa^2+3920}{179200} n^{-\frac{3}{2}} + \frac{5\kappa^4+96\kappa^2+640}{7680\kappa^{\frac{8}{3}}} n^{-\frac{11}{6}} \\ & - \frac{823\kappa^6+647200\kappa^4-2464000\kappa^2-25088000}{258048000\kappa^{\frac{10}{3}}} n^{-\frac{13}{6}} \\ & + \frac{3064+33\kappa^2}{716800} n^{-\frac{5}{2}} \Bigg), \quad n \to \infty. \end{aligned} \tag{61}$$

3.2 Case II: $k = O\left(\frac{n}{2}\right)$

We now set

$$k = \left\lfloor \frac{n}{2} \right\rfloor + 1 - j = \frac{n}{2} - \alpha + 1 - j, \tag{62}$$

where $\alpha = \mathrm{frac}\left(\frac{n}{2}\right)$ (the fractional part of $\frac{n}{2}$) and $j = 0, 1, 2, \ldots$. Using (62) and

$$\tau_k^n = \sum_{i \geq 1} b_i\left(j\right) n^{-i} \tag{63}$$

in (56) we obtain, as $n \to \infty$

$$\begin{aligned} & b_1 = \xi, \quad b_2 = -\frac{\xi}{4}, \quad b_3 = \frac{\xi}{48}\left(3+16\xi^2\right), \\ & b_4 = -\frac{\xi}{192}\left(3+64\xi^2\right), \quad b_5 = \frac{\xi}{3840}\left(15+800\xi^2+1024\xi^4\right), \\ & b_6 = -\frac{\xi}{15360}\left(15+1600\xi^2+7424\xi^4\right), \end{aligned} \tag{64}$$

with

$$\xi\left(j\right) = \frac{\pi}{4}(2j+2\alpha-1). \tag{65}$$

Using (63)-(64) in (55), we obtain

$$\zeta_k^n \sim \sqrt{2}\xi\left(n^{-\frac{1}{2}} - \frac{1}{4} n^{-\frac{3}{2}} + \frac{3+8\xi^2}{48} n^{-\frac{5}{2}} - \frac{3+80\xi^2}{192} n^{-\frac{7}{2}} \right), \quad n \to \infty. \tag{66}$$

In Table1 we compare the exact value of the positive zeros of $H_{20}\left(x\right)$ with the approximations given by solving (56) numerically and formulas (61) and

Table 1: A comparison of the exact and approximate values for the positive zeros of $H_{20}(x)$.

ζ_k^n	(56)	(66)	(61)
.24534	.24536	.24536	_
.73747	.73751	.73750	_
1.2341	1.2342	1.2340	_
1.7385	1.7387	1.7376	_
2.2550	2.2552	2.2512	2.2592
2.7888	2.7892	2.7779	2.7912
3.3479	3.3486	_	3.3492
3.9448	3.9456	_	3.9460
4.6037	4.6056	_	4.6055
5.3875	5.3939	_	5.3937

(66). Note that the biggest error corresponds to the larger zero, where the asymptotic approximation (54) almost breaks down.

We summarize the results of this section in the following theorem.

Theorem 6 *Letting $\zeta_1^n > \zeta_2^n > \cdots > \zeta_n^n$ be the zeros of $H_n(x)$, enumerated in decreasing order, we have:*

1.
$$\zeta_k^n \sim \sqrt{2}\left(n^{\frac{1}{2}} - \frac{\kappa^{\frac{2}{3}}}{8}n^{-\frac{1}{6}} + \frac{1}{4}n^{-\frac{1}{2}} - \frac{\kappa^2+80}{640\kappa^{\frac{2}{3}}}n^{-\frac{5}{6}}\right), \quad n \to \infty,$$

   where $k = O(1)$ and $\kappa(k)$ was defined in (60).

2.
$$\zeta_k^n \sim \sqrt{2}\xi\left(n^{-\frac{1}{2}} - \frac{1}{4}n^{-\frac{3}{2}} + \frac{3+8\xi^2}{48}n^{-\frac{5}{2}} - \frac{3+80\xi^2}{192}n^{-\frac{7}{2}}\right), \quad n \to \infty,$$

   where $k = \frac{n}{2} - \alpha + 1 - j$, $\alpha = \mathrm{frac}\left(\frac{n}{2}\right)$ and $\xi(j)$ was defined in (65).

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